Exam 1.

Put option:

Set $\bar{p} = \frac{\hat{p}}{1+r}$, $\bar{q} = \frac{\hat{q}}{1+r}$, then

$$\begin{cases} 10 = 12\bar{p} + 8\bar{q} \\ 1.13 = 2\bar{q} \end{cases} \Rightarrow \begin{cases} \bar{p} = 0.457 \\ \bar{q} = 0.565. \end{cases}$$

$$\bar{p} + \bar{q} = \frac{1}{1+r} = 1.022 \Rightarrow 1+r = 0.4785.$$

For $\hat{p}, \hat{q}$,

$$\begin{cases} 10 = 12\hat{p} + 6\hat{q} \\ \hat{p} + \hat{q} = 1.022 \end{cases} \Rightarrow V_0 = 4\hat{q} = 1.51$$

Call option:

$$\begin{cases} S_0 = \tilde{p} \cdot \frac{S_1(H)}{1+r} + \tilde{q} \cdot \frac{S_1(T)}{1+r} \\ V_0 = \tilde{p} \cdot \frac{V_1(H)}{1+r} + \tilde{q} \cdot \frac{V_1(T)}{1+r} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} S_0 = \bar{p} \cdot S_1(H) + \bar{q} \cdot S_1(T) \\ V_0 = \bar{p} \cdot V_1(H) \end{cases}$$

$$\Downarrow$$

$$\begin{cases} 10 = \bar{p} \cdot 12 + \bar{q} \cdot 8 \\ 1.13 = 2\bar{q}. \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \bar{p} = 0.5651 \\ \bar{q} = 0.4025. \end{cases}$$

$$\bar{p} + \bar{q} = \frac{1}{1+r} = 0.9675.$$

$$\Rightarrow \begin{cases} 10 = 12\hat{p} + 6\hat{q} \\ \hat{p} + \hat{q} = 0.965 \end{cases} \Rightarrow 2\hat{p} = 1.39833$$

Exam 2.

① $E_n[f_{n+1}(S_{n+1}, Y_{n+1})] = pf(uS_n, Y_n + uS_n) + qf(dS_n + Y_n + dS_n) = g(S_n, Y_n)$.

$g(s, y) = pf(us, y + us) + qf(ds + y + ds)$

② Connect f_n with f_{n+1},

$$f_n(S_n, Y_n) = V_n = \frac{1}{1+r}\tilde{E}_n[V_{n+1}] = \frac{1}{1+r}\tilde{E}_n[f_{n+1}(S_{n+1}, Y_{n+1})]$$

where

$$\tilde{E}_n[f_{n+1}(S_{n+1}, y_{n+1})] = \tilde{p} f_{n+1}(uS_n, Y_n + uS_n) + \tilde{q} f_{n+1}(dS_n + y_n + dS_n)$$

then
$$\begin{cases} f_N(s, y) = \max\left(\frac{y}{N+1} - K, 0\right) \\ f_n(s, y) = \frac{1}{1+r}[\tilde{p} f_{n+1}(us, y + us) + \tilde{q} f_{n+1}(ds, y + ds)] \end{cases}$$

Exam 3.

Theorem 2.4.5, Corollary 2.4.6.

① $\frac{X_n}{(1+r)^n}$ is martingale under $\hat{p}$, then $\hat{E}\frac{X_N}{(1+r)^N} = X_0$

② Set $X_n = \hat{E}_n\left(\frac{X}{(1+r)^{N-n}}\right)$ and $X_0 = \tilde{E}\left[\frac{X}{(1+r)^N}\right]$

$X_{n+1} = \Delta_n^* S_{n+1}$

and $\Delta_0^*, \cdots, \Delta_{N-1}^*$ assumed as $\Delta_n^* = \frac{X_{n+1}(H) - X_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)}$, then it follows $+ (1+r)(X_n - \Delta_n^* S_n)$.